COHOMOLOGY FOR NORMAL SPACES

By

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To

Pat, Mom, and Dad

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INTRODUCTION

This paper presents a definition of cohomology groups of a topological space relative to a subset of the space. The definition employed was suggested to the author by A.D. Wallace, and is a modification of an earlier definition, also due to Wallace, which was exploited by Spanier in [7]. Both definitions, as will be seen below, involve the notion of p-functions and hence have their roots in the works of Alexander and Kolmogoroff. These definitions agree on compact Hausdorff spaces and, after a suitable shift in dimension, yield groups isomorphic to those in [6].

If X is a topological space we let X^{p+1} denote the cartesian product of X with itself p+1 times and define $C^p(X) = \{\varphi \mid \varphi: X^{p+1} \longrightarrow G\}$, where G is a fixed, though arbitrary, abelian group. Then $C^p(X)$ is itself an abelian group, if addition of two elements in $C^p(X)$ is defined pointwise; this group is called the group of p-functions. If $\mathcal{U}$ is an open covering of X, we set $\mathcal{U}^{(p+1)} = \cup\{ u^{p+1} \mid u \in \mathcal{U}\}$. Then for each subset A of X, and each integer $p \geq 0$, we may define $C^p(X,A) = \{\varphi \mid \varphi \in C^p(X)$ and there exists an open cover $\mathcal{U}$ of A such that $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}\}$.

For each integer $p \geq 0$ there is a homomorphism, defined in Chapter I, $\bar{\delta}: C^p(X) \longrightarrow C^{p+1}(X)$ having the properties that $\bar{\delta}\bar{\delta} = 0$ and $\bar{\delta}[C^p(X,X)]$ is contained in $C^{p+1}(X,X)$. Then other subgroups of $C^p(X)$ may be defined by $Z^p(X,A) = C^p(X,A) \cap \bar{\delta}^{-1}[C^{p+1}(X,X)]$; $B^p(X,A) = \bar{\delta}[C^{p-1}(X,A)] + C^p(X,X)$ (for $p = 0$, $B^p(X,A) = \{0\}$). The definition used by Spanier of the p-th cohomology group of the space X relative to the subset A, denoted by $H^p(X,A)$, is the quotient group $Z^p(X,A) / B^p(X,A)$.

Our departure from this definition is effected by redefining $C^p(X,A)$ as the set of p-functions, φ, for which there exists a finite open cover $\mathcal{U}$ of A such that $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$. A similar distinction is found between the Čech cohomology theory based on finite open coverings and the Čech theory, advanced by Dowker, based on pairs of infinite coverings.

Spanier showed that the theory developed in [7] was a cohomology theory, in the sense of Eilenburg and Steenrod [3], on the category of compact pairs. This result then, carries over to the development presented in this paper. In fact, most of the axioms of [3] will be verified for general topological pairs (the only exception being the Homotopy Axiom). Each time an axiom is verified the axiom will be identified by a parenthetical insertion referring to the axiom exactly as it is numbered on page fourteen of [3].

In Chapter I we review some of Spanier's results

and definitions for use throughout this paper.

Chapter II presents our basic definitions and major results. The development follows closely that of Wallace's notes on Algebraic Topology [8]. Many of the theorems in [8] are proved under the hypothesis that the topological space in question is fully normal. Virtually all of these same theorems, including the Reduction and Extension Theorems, are proved for normal spaces. We employ the resulting generality to show that a connected, normal, T_1 space with trivial first cohomology group is unicoherent.

In Chapter III we disprove a conjecture that a particular group assignment, defined for a special class of spaces, will assign the trivial group to spaces which are not locally connected.

We conclude with Chapter IV by discussing related subjects (e.g. codimension) and open questions.

CHAPTER I

PRELIMINARY RESULTS AND DEFINITIONS

We review some of Spanier's results which will be needed in the sequel. We assume throughout this paper that G is a fixed, though arbitrary, abelian group. The term " mapping" will be used to mean " continuous function" and X^{p+1} will denote the cartesian product of the topological space X with itself p+1 times.

1.1 <u>Definition</u>: Let X be a topological space and let $p \geq 0$ be any integer. Then $C^p(X) = \{\varphi | \varphi: X^{p+1} \longrightarrow G\}$. For each pair $\varphi, \psi \in C^p(X)$ define $(\varphi+\psi): X^{p+1} \longrightarrow G$ by $(\varphi+\psi)(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p) + \psi(x_0,\ldots,x_p)$.

1.2 <u>Definition</u>: 1) For any set P we let the diagonal $D(P^n)$ of P^n be $\cup\{\{x\}^n | x \in P\}$.

2) If $f: X \longrightarrow Y$ is a function from a topological space X to a topological space Y, define $f^{\#}: C^p(Y) \longrightarrow C^p(X)$ by $[f^{\#}(\varphi)](x_0,\ldots,x_p) = \varphi[f(x_0),\ldots,f(x_p)]$.

1.3 <u>Definition</u>: We define $\bar{\delta}: C^p(X) \longrightarrow C^{p+1}(X)$ by $[\bar{\delta}(\varphi)](x_0,\ldots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i \varphi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1})$

where $(x_0,..,\hat{x}_i,..,x_{p+1}) = (x_0,..,x_{i-1}, x_{i+1},..,x_{p+1}) \in X^{p+1}$.

1.4 <u>Definition</u>: Let $f,g: X \longrightarrow Y$. Then define for $p > 0$, $D: C^p(Y) \longrightarrow C^{p-1}(X)$ by $[D(\varphi)](x_0,\ldots,x_{p-1}) = \sum_{i=0}^{p-1}(-1)^i \varphi[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_{p-1})]$. The properties of $C^p(X)$, $f^\#$, $\bar{\delta}$, and D, of which we will constantly make use, are collected in the following:

1.5 <u>Theorem</u>: 1) $C^p(X)$ is an abelian group

2) $f^\#$ and $\bar{\delta}$ are homomorphisms

3) $f^\#\bar{\delta} = \bar{\delta}f^\#$

4) $\bar{\delta}\bar{\delta} = 0$

5) $D\bar{\delta} + \bar{\delta}D = f^\# - g^\#$ if $p \geq 1$

6) $D\bar{\delta} = f^\# - g^\#$ if $p = 0$

CHAPTER II

THE COHOMOLOGY GROUPS OF A SPACE MODULO A SUBSET

We shall adhere to the definitions and notation of Chapter I, and shall introduce new definitions, conventions and algebraic lemmas as they are needed. We omit the proofs for these lemmas if they are available in standard texts.

2.1 Notation: If $\mathcal{U}$ is a family of sets $\mathcal{U}^{(p)} = \cup\{A^p | A \in \mathcal{U}\}$. If f: $X \longrightarrow Y$ is a function from a space X to a space Y, if $A \subset X$ and if $f(A) \subset B \subset Y$, then we write $f: (X,A) \longrightarrow (Y,B)$.

2.2 Definition: If $A \subset X$ we define $C^p(X,A) = \{\varphi | \varphi \in C^p(X)$ and $\exists$ a finite open cover $\mathcal{U}$ of $A \ni \varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}\}$. We often write $C_0^p(X) = C_0^p(X,A) = C^p(X,X)$.

2.3 Lemma: 1) $C^p(X,A)$ is a subgroup of $C^p(X)$

2) If $f: (X,A) \longrightarrow (Y,B)$, f continuous, then $f^{\#}[C^p(Y,B)] \subset C^p(X,A)$.

3) If $A \subset X$, then $\delta[C^p(X,A)] \subset C^{p+1}(X,A)$.

Proof: 1) Let φ, $\psi \in C^p(X,A)$ and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of $A \ni \varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$ and $\psi = 0$ on $\mathcal{V}^{(p+1)} \cap A^{p+1}$, then $\mathcal{W} = \{u \cap v | u \in \mathcal{U}$ and $v \in \mathcal{V}\}$ is a

finite open cover of A such that $\varphi - \psi = 0$ on $\mathcal{W}^{(p+1)} \cap A^{p+1}$. Thus $\varphi - \psi \in C^p(X,A)$.

2) Let $\varphi \in C^p(Y,B)$, then $\exists$ a finite open cover $\mathcal{U}$ of B such that $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap B^{p+1}$. Then $\mathcal{V} = \{f^{-1}(u) \mid u \in \mathcal{U}\}$ is a finite open cover of A such that $f^{\#}(\varphi) = 0$ on $\mathcal{V}^{(p+1)} \cap A^{p+1}$. Thus $f^{\#}(\varphi) \in C^p(X,A)$.

3) Let $\varphi \in C^p(X,A)$, then $\exists$ a finite open cover $\mathcal{U}$ of A such that $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$, and one easily checks that $\bar{\delta}\varphi = 0$ on $\mathcal{U}^{(p+2)} \cap A^{p+2}$, hence $\bar{\delta}\varphi \in C^{p+1}(X,A)$.

2.4 Definition: Let A be a subset of a space X. Then

1) $Z^p(X,A) = C^p(X,A) \cap \bar{\delta}^{-1}[C_0^{p+1}(X,A)]$

2) $B^p(X,A) = \begin{cases} \{0\}, & p = 0 \\ C_0^p(X,A) + \bar{\delta}[C^{p-1}(X,A)]: & p \geq 1. \end{cases}$

2.5 Lemma: 1) $Z^p(X,A)$ is a subgroup of $C^p(X,A)$

2) $B \subset A \subset X \longrightarrow C^p(X,A) \subset C^p(X,B)$: hence $Z^p(X,A) \subset Z^p(X,B)$.

3) $B^p(X,A)$ is a subgroup of $Z^p(X,A)$

4) If A is a closed subset of X and $\varphi \in Z^p(X,A)$, then $\exists$ a finite open cover $\mathcal{U}$ of X with (i) $\bar{\delta}\varphi = 0$ on $\mathcal{U}^{(p+2)}$ (ii) $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$.

Proof: 1) and 2) are clear. For 3) we recall that $\bar{\delta}\bar{\delta} = 0$ and use Lemma 2.3, part three, to establish the appropriate inclusions. For 4) assume $\varphi \in Z^p(X,A)$, then $\varphi \in C^p(X,A)$ and $\bar{\delta}\varphi \in C_0^{p+1}(X)$. Hence $\exists$ a finite open cover $\mathcal{U}$ of A $\ni \varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$ and a finite open cover

$\mathcal{V}$ of $X \ni \bar{\delta}\varphi = 0$ on $\mathcal{V}^{(p+2)}$. Let $\mathcal{O} = \mathcal{U} \cup \{X-A\}$ and $\mathcal{W} = \{v \cap o \mid v \in \mathcal{V}$ and $o \in \mathcal{O}\}$, then $\mathcal{W}$ is a finite open cover of X satisfying (i) and (ii).

2.6 <u>Definition</u>: If A is a subset of a space X, we define $H^p(X,A) = Z^p(X,A) / B^p(X,A)$.

2.7 <u>Lemma</u>: If $f\colon (X,A) \longrightarrow (Y,B)$ is continuous, then

1) $f^{\#}[Z^p(Y,B)] \subset Z^p(X,A)$

2) $f^{\#}[B^p(Y,B)] \subset Z^p(X,A)$.

<u>Proof</u>: 1) If $\varphi \in Z^p(Y,B)$, then $\varphi \in C^p(Y,B)$ and $\bar{\delta}\varphi \in C_0^{p+1}(Y,B)$, hence $f^{\#}(\varphi) \in C^p(X,A)$ and $\bar{\delta}[f^{\#}(\varphi)] = f^{\#}[\bar{\delta}\varphi]$ $C_0^{p+1}(X)$. Thus $f^{\#}(\varphi) \in Z^p(X,A)$.

2) If $\varphi \in B^p(Y,B)$, then $\varphi = \psi + \bar{\delta}\theta$ where ψ $C_0^p(Y)$ and $\theta \in C^{p-1}(Y,B)$. Hence $f^{\#}(\varphi) = f^{\#}(\psi) + f^{\#}(\bar{\delta}\theta) = f^{\#}(\psi) + \bar{\delta}[f^{\#}(\theta)] \in C_0^p(X) + \bar{\delta}[C^{p-1}(X,A)] = B^p(X,A)$, $p \geq 1$. The case $p = 0$ is clear.

2.7 <u>Induced Homomorphism Theorem</u>: Let P be a group with subgroup P_0, let Q be a group with subgroup Q_0 and let f be a homomorphism of P into Q such that $f(P_0) \subset Q_0$. Then if $\alpha\colon P \longrightarrow P/P_0$ and $\beta\colon Q \longrightarrow Q/Q_0$ are the natural homomorphisms, there exists one and only one homomorphism g such that $g\alpha = \beta f$.

2.8 <u>Theorem</u>: Let $f\colon (X,A) \longrightarrow (Y,B)$ be continuous, let $\alpha\colon Z^p(X,A) \longrightarrow H^p(X,A)$ and $\beta\colon Z^p(Y,B) \longrightarrow H^p(Y,B)$ be the natural homomorphisms and define $f_0^{\#}\colon Z^p(Y,B) \longrightarrow Z^p(X,A)$ by $f_0^{\#}(\varphi) = f^{\#}(\varphi)$ for each $\varphi \in Z^p(Y,B)$. Then $\exists$ a unique homomor-

phism f^*: $H^p(Y,B) \longrightarrow H^p(X,A)$ such that $\alpha f_0^* = \beta f^*$.

Proof: Induced Homomorphism Theorem.

2.9 Theorem: $\varphi \in Z^0(X)$ if and only if $\varphi: X \longrightarrow G$ is continuous in the discrete topology of G, and $\{\varphi^{-1}(g) \mid g \in G\}$ is finite.

Proof: If $\varphi \in Z^0(X)$ then $\exists$ a finite open cover of X such that $\bar{\delta}\varphi = 0$ on $\mathcal{U}^{(2)}$. Let $\varphi(x) = g \in G$, then $\exists$ $u \in \mathcal{U}$ with $x \in u$. Now if $y \in u$ then $\bar{\delta}\varphi(x,y) = \varphi(y) - \varphi(x) = 0$. Thus $\varphi(y) = \varphi(x) = g$ and $u \subset \varphi^{-1}(g)$. Hence φ is continuous. Set $\mathcal{U} = \{u_i \mid 1 \leq i \leq n\}$ and let $g_i \in G$ be such that $u_i \subset \varphi^{-1}(g_i)$, then $X = \cup\{u_i \mid 1 \leq i \leq n\} \subset \cup\{\varphi^{-1}(g_i) \mid 1 \leq i \leq n\}$, thus if $x \in \varphi^{-1}(g)$ for some $g \in G$, then $\exists$ an integer k between 1 and n with $x \in \varphi^{-1}(g_k)$. Hence $\varphi^{-1}(g) = \varphi^{-1}(g_k)$ and we have $\{\varphi^{-1}(g) \mid g \in G\} \subset \{\varphi^{-1}(g_i) \mid 1 \leq i \leq n\} \cup \{\emptyset\}$. Suppose now that $\varphi: X \longrightarrow G$ is continuous in the discrete topology of G and that $\{\varphi^{-1}(g) \mid g \in G\}$ is finite. Then $\exists$ $g \in G$ such that $x \in G$ such that $x \in \varphi^{-1}(g)$, and if $y \in \varphi^{-1}(g)$ we have $\bar{\delta}\varphi(x,y) = \varphi(y) - \varphi(x) = g - g = 0$. Hence $\bar{\delta}\varphi \in C_0^1(X)$ and $\varphi \in Z^0(X)$.

2.10 Theorem: If $G \neq \{0\}$, then X is connected if and only if $H^p(X,x) = \{0\}$ for each (for some) $x \in X$.

Proof: Let X be connected, then if $\varphi \in Z^0(X,x) \cong H^0(X,x)$, $\varphi \in C^0(X,x) \cap \bar{\delta}^{-1}[C_0^1(X)] \subset C^0(X) \cap \bar{\delta}^{-1}[C_0^1(X)] = Z^0(X)$. Thus φ is continuous in the discrete topology of G and is therefore a constant function. Hence $\varphi(x) = 0 \longrightarrow \varphi(X) = 0$, and we have $\{0\} = Z^0(X,x) \cong H^0(X,x)$. Now assume

X is the union of two disjoint open sets A and B. Fix x in X and assume $x \in A$. Define $\varphi: X \longrightarrow G$ by $\varphi(A) = 0$ and $\varphi(B) = g \neq 0$. Then φ is continuous in the discrete topology of G and $\{\varphi^{-1}(g) \mid g \in G\} = \{A,B\}$ is finite. Thus $\varphi \in Z^0(X)$, but $x \in A$ open $\longrightarrow \varphi \in C^0(X,x)$, so we have $0 \neq \varphi \in Z^0(X,x) \cong H^0(X,x)$, a contradiction.

Conventions: If A is a subset of a set B and if $f: A \longrightarrow B$ is defined by $f(x) = x$ for each x in A, then f is called the <u>inclusion map</u> of A into B and is denoted by $f: A \subset B$.

If $f: G \longrightarrow H$ is a homomorphism from the group G into the group H, then $\{h \mid h \in H$ and $h = f(g)$ for some $g \in G\}$, the <u>image</u> of f, will be denoted by $I(f)$ and $\{g \mid f(g) = 0\}$, the <u>kernel</u> of f, will be denoted by $K(f)$.

2.11 <u>Theorem</u>: Let X be a space, let $B \subset A \subset X$, let $\gamma: Z^p(A,B) \longrightarrow H^p(A,B)$ and $\alpha: Z^{p+1}(X,A) \longrightarrow H^{p+1}(X,A)$ be the natural homomorphisms and let $t: A \subset X$. Then

1) For each $h \in H^p(A,B)$ there exists such a $\varphi \in C^p(X)$ that $\gamma t^{\#}(\varphi) = h$ and $\bar{\delta}\varphi \in Z^{p+1}(X,A)$.

2) If $\varphi, \psi \in C^p(X)$, if $t^{\#}(\varphi), t^{\#}(\psi) \in Z^p(A,B)$ and if $\gamma t^{\#}(\varphi) = \gamma t^{\#}(\psi)$, then $\bar{\delta}(\varphi - \psi) \in B^{p+1}(X,A)$.

3) $\delta = \alpha \bar{\delta} t^{\#-1} \gamma^{-1}$ is a homomorphism from $H^p(A,B)$ into $H^{p+1}(X,A)$.

<u>Proof:</u> We first note that if $\psi \in C^p(X)$ with $t^{\#}(\psi) = \varphi \in C_0^p(A)$ then $\psi \in C^p(X,A)$. For $\exists$ a finite cover of A by sets open in A, $\mathcal{U}$, with $\varphi = 0$ on $\mathcal{U}^{(p+1)}$, hence if we

write $\mathcal{U} = \{v_i \cap A | 1 \leq i \leq n\}$ where v_i is open in X and $\mathcal{V} = \{v_i | 1 \leq i \leq n\}$, we have that $\mathcal{V}$ is a finite open cover of $A \ni \psi = 0$ on $\mathcal{V}^{(p+1)} \cap A^{p+1}$.

1) Let $h \in H^p(A,B)$, then $\exists\ \varphi \in Z^p(A,B)$ with $\gamma(\varphi) = h$. Define $\psi: X^{p+1} \longrightarrow G$ by the equations $\psi(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p)$ if $(x_0,\ldots,x_p) \in A^{p+1}$; $\psi(x_0,\ldots,x_p) = 0$ if $(x_0,\ldots,x_p) \in X^{p+1} - A^{p+1}$. Clearly $t^{\#}(\psi) = \varphi$, hence $\gamma[t^{\#}(\psi)] = \gamma(\varphi) = h$. Now $\bar{\delta}\varphi \in C_0^{p+1}(A)$ and $t^{\#}[\bar{\delta}\psi] = \bar{\delta}[t^{\#}(\psi)] = \bar{\delta}\varphi$ hence $\bar{\delta}\psi \in C^{p+1}(X,A)$.

2) Let $\varphi, \psi \in C^p(X)$; $t^{\#}(\varphi), t^{\#}(\psi) \in Z^p(A,B)$ with $\gamma t^{\#}(\varphi) = \gamma t^{\#}(\psi)$, then $t^{\#}(\varphi-\psi) \in B^p(A,B)$ and hence $t^{\#}(\varphi-\psi) = \varphi_1 + \bar{\delta}(\varphi_2)$ where $\varphi_1 \in C_0^p(A)$ and $\varphi_2 \in C^{p-1}(A,B)$, $p \geq 1$. Let $\varphi_2' \in t^{-1}(\varphi_2) \subset C^{p-1}(X)$ and define $\varphi_1' = \varphi_1$ on A^{p+1}; $\varphi_1' = \varphi - \psi - \bar{\delta}(\varphi_2')$ on $X^{p+1} - A^{p+1}$. Then $t^{\#}(\varphi_1') = \varphi_1$, hence $t^{\#}(\varphi-\psi) - \varphi_1 = t^{\#}(\varphi-\psi) - t^{\#}(\varphi_1') = t^{\#}[(\varphi-\psi) - \varphi_1'] = \bar{\delta}(\varphi_2) = \bar{\delta}[t^{\#}(\varphi_2')] = t^{\#}[\bar{\delta}(\varphi_2')]$, hence $(\varphi-\psi) - \varphi_1' = \bar{\delta}(\varphi_2')$ on all X^{p+1}. Thus $\bar{\delta}(\varphi-\psi) - \bar{\delta}(\varphi_1') = \bar{\delta}\bar{\delta}(\varphi_2') = 0$, from which $\bar{\delta}(\varphi-\psi) = \bar{\delta}(\varphi_1')$; but $t^{\#}(\varphi_1') = \varphi_1 \in C_0^p(A) \longrightarrow \varphi_1' \in C^p(X,A)$. Thus $\bar{\delta}(\varphi_1') \in \bar{\delta}[C^p(X,A)]$ and $\bar{\delta}(\varphi-\psi) = \bar{\delta}(\varphi_1') \in B^p(X,A)$ The case $p = 0$ is trivial.

3) δ is well defined is immediate from 1) and 2).

2.12 <u>Definition</u>: A sequence of homomorphisms

$$A_0 \xrightarrow{h_0} A_1 \xrightarrow{h_1} \cdots \xrightarrow{h_{n-1}} A_n \xrightarrow{h_n} A_{n-1} \longrightarrow \cdots$$

is exact if and only if h_0 is a monomorphism and $I(h_i) = K(h_{i+1})$ for $i \geq 0$.

<u>Notation</u>: The following notations will hold through

Theorem 2.16: $B \subset A \subset X$, $j: (X,B) \subset (X,A)$, $i: (A,B) \subset (X,B)$, and $t: (A,\square) \subset (X,\square)$. $\alpha: Z^p(X,A) \longrightarrow H^p(X,A)$, $\beta: Z^p(X,B) \longrightarrow H^p(X,B)$, and $\gamma: Z^p(A,B) \longrightarrow H^p(A,B)$ are the natural homomorphisms.

2.13 <u>Lemma</u>: 1) $j^*: H^0(X,A) \longrightarrow H^0(X,B)$ is a monomorphism.

2) $I(j^*) = K(i^*)$ for $p \geq 0$.

<u>Proof</u>: 1) Let $\varphi \in Z^0(X,A) \cong H^0(X,A)$ and suppose $j^\#\varphi(x) = 0$ for each x in X, then $\varphi[j(x)] = \varphi(x) = 0$ for each x in X. Thus $j^*\alpha(\varphi) = 0 \longrightarrow \beta j^\#(\varphi) = 0 \longrightarrow j^\#\varphi = 0 \longrightarrow \varphi = 0 \longrightarrow \alpha\varphi = 0 \longrightarrow j^*$ is a monomorphism.

2) We first show that $I(j^*) \subset K(i^*)$. If $\varphi \in Z^p(X,A)$ then $\exists$ a finite open cover $\mathcal{U}$ of $A \ni \varphi = 0$ on $\mathcal{U}^{(p+1)} \cap A^{p+1}$. Let $\mathcal{V} = \{u \cap A \mid u \in \mathcal{U}\}$, then $\mathcal{V}$ is a finite open cover of A with $i^\# j^\#\varphi = 0$ on $\mathcal{V}^{(p+1)}$. Hence $i^\# j^\#\varphi \in C_0^p(A) \subset B^p(A,B)$. Since the natural homomorphisms are onto the proof is complete. To prove $K(i^*) \subset I(j^*)$ we suppose $i^\#\varphi \in B^p(A,B)$, then $i^\#\varphi = \varphi_1 + \bar{\delta}(\varphi_2)$ where $\varphi_1 \in C_0^p(A)$ and $\varphi_2 \in C^{p-1}(A,B)$, $p \geq 1$. Let $\theta \in C^{p-1}(X,B)$ with $i^\#\theta = \varphi_2$ and define $\psi: X^{p+1} \longrightarrow G$ by $\psi = \varphi_1$ on A^{p+1} and $\psi = \varphi - \bar{\delta}(\theta)$ on $X^{p+1} - A^{p+1}$. Then $\psi \in C^p(X,A)$ and if $(x_0,\ldots,x_p)$ is in A^{p+1} we have $\psi(x_0,\ldots,x_p) = \varphi_1(x_0,\ldots,x_p)$ $= i^\#\varphi(x_0,\ldots,x_p) - \bar{\delta}\varphi_2(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p) - \bar{\delta}i^\#\theta(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p) - i^\#\bar{\delta}\varphi(x_0,\ldots,x_p) = \varphi(x_0,\ldots,x_p) - \bar{\delta}\varphi(x_0,\ldots,x_p)$. Thus $\bar{\delta}[\varphi - \bar{\delta}\theta] = \bar{\delta}\varphi - \bar{\delta}\bar{\delta}\theta = \bar{\delta}\varphi \in C_0^{p+1}(X)$. Hence $\psi \in Z^p(X,A)$ and $\varphi - j^\#\psi = \bar{\delta}\theta \in \bar{\delta}[C^{p-1}(X,B)] \subset B^p(X,B)$. Again the case $p = 0$ is trivial

and inclusion follows from the fact that the natural homomorphisms are onto.

2.14 Lemma: $I(i^*) = K(\delta)$

Proof: $I(i^*) \subset K(\delta)$. Let $\varphi \in Z^p(X,B)$, then $\varphi \in t^{\#-1}i^{\#}\varphi$ hence $\bar{\delta}t^{\#-1}i^{\#}\varphi = \bar{\delta}\varphi$. But $\varphi \in \bar{\delta}^{-1}[C_0^{p+1}(X)]$, so $\bar{\delta}\varphi \in C^{p+1}(X)$, a subset of $B^{p+1}(X,A)$. The inclusion is then clear. To show $K(\delta) \subset I(i^*)$ we suppose $\varphi \in Z^p(A,B)$, $\psi \in t^{\#-1}(\varphi)$ with $\bar{\delta}\psi \in B^{p+1}(X,A)$. Then $\bar{\delta}\psi = \varphi_1 + \bar{\delta}(\varphi_2)$ where $\varphi_1 \in C_0^{p+1}(X)$ and $\varphi_2 \in C^p(X,A)$. Define $\theta = \psi - \varphi_2$, then $\bar{\delta}\theta = \varphi_1 \in C_0^{p+1}(X)$ hence $\theta \in \bar{\delta}^{-1}[C_0^{p+1}[X)]$. Also $t^{\#}\psi = \varphi \in C^p(A,B)$ and hence $\psi \in C^p(X,B)$; therefore $\theta \in C^p(X,B)$ and we have $\theta \in Z^p(X,B)$. Now $\varphi - i^{\#}\theta = \varphi - i^{\#}\psi - i^{\#}\varphi_2 = \varphi - \varphi - i^{\#}\varphi_2 = -i^{\#}\varphi_2 \in C_0^p(A) \subset B^p(A,B)$.

2.15 Lemma: $I(\delta) = K(j^*)$

Proof: $I(\delta) \subset K(j^*)$. Let $\psi \in t^{\#-1}\varphi$ with $\varphi \in Z^p(A,B)$, then $t^{\#}\psi = \varphi \in C^p(A,B) \longrightarrow \psi \in C^p(X,B)$. Hence $\bar{\delta}\psi \in \bar{\delta}[C^p(X,B)] \subset B^{p+1}(X,B)$ and $j^{\#}\bar{\delta}\psi = \bar{\delta}\psi$. To complete the proof we appeal to the natural homomorphisms. To show $K(j^*) \subset I(\delta)$ we let $\varphi \in Z^{p+1}(X,A)$ with $i^{\#}\varphi \in B^{p+1}(X,B)$. Then $\varphi = j^{\#}\varphi = \varphi_1 + \bar{\delta}(\varphi_2)$ where $\varphi_1 \in C_0^{p+1}(X)$ and $\varphi_2 \in C^p(X,B)$. Thus $i^{\#}\varphi_2 \in C^p(A,B)$ and $\bar{\delta}i^{\#}(\varphi_2) = i^{\#}\bar{\delta}(\varphi_2) = i^{\#}\varphi - i^{\#}\varphi_1 \in C_0^{p+1}(A)$. Hence $i^{\#}\varphi_2 \in Z^p(A,B)$ and $\varphi_2 \in t^{\#-1}i\ (\varphi_2)$, so $\bar{\delta}t^{\#-1}i^{\#}(\varphi_2) = \bar{\delta}(\varphi_2) \in C^{p+1}(X,B)$ and $\varphi - \bar{\delta}(\varphi_2) = \varphi_1 \in C_0^{p+1}(X) \subset B^{p+1}(X,A)$.

2.16 Theorem: $H^0(X,A) \xrightarrow{j^*} H^0(X,B) \xrightarrow{i^*} H^0(A,B) \xrightarrow{\delta} H^1(X,A) \xrightarrow{j^*} \ldots \xrightarrow{\delta} H^n(X,A) \xrightarrow{j^*} H^n(X,B) \xrightarrow{i^*}$

$H^n(A,B) \xrightarrow{\delta} H^{n+1}(X,A) \xrightarrow{j^*} \ldots$ is exact. (Axiom 4 c).

Proof: The three previous Lemmas.

2.17 Corollary: $H^p(X,X) = 0$, for any space X and any $p \geq 0$.

Proof: One takes $B = A = X$ in the previous theorem and easily verifies that i^*j^* is the identity function, as well as the zero function, on $H^p(X,X)$.

2.18 Corollary: If A is a connected subset of a space X or if $A = \square$, then $\delta\colon H^0(A) \longrightarrow H^1(X,A)$ is the zero function.

Proof: Recall that $H^0(A) \cong Z^0(A)$ and $H^0(X) \cong Z^0(X)$. Each $\varphi \in Z^0(A)$ is continuous in the discrete topology of G and hence is a constant function, since A is connected. The constant function $\psi\colon X \longrightarrow G$ defined by extending φ to all X is such that $i^\#\psi = \varphi$, where we have taken $B = \square$. Thus $i^\#$, and consequently i^*, is an epimorphism and hence $H^0(A) = I(i^*) = K(\delta)$ if A is not empty. But $A = \square$ is clear.

2.19 Theorem: 1) If $f\colon X \longrightarrow Y$ and if $g\colon Y \longrightarrow Z$, then $(gf)^\# = f^\# g^\#$.

2) If $f\colon (X,A) \longrightarrow (Y,B)$, if $g\colon (Y,B) \longrightarrow (Z,C)$, and if f and g are continuous, then $(gf)^* = f^*g^*$. (Axiom 2 c).

Proof: 1) $[(gf)^\#\varphi](x_0,\ldots,x_p) = \varphi(gf[x_0],\ldots,gf[x_p]) = g^\#\varphi(f[x_0],\ldots,f[x_p]) = [f^\#(g^\#\varphi)](x_0,\ldots,x_p) = [f^\# g^\#]\varphi(x_0,\ldots,x_p)$. Thus $(gf)^\#\varphi = [f^\# g^\#]\,\varphi$ if $\varphi \in C^p(X)$.

2) Let $\alpha: Z^p(X,A) \longrightarrow H^p(X,A)$, $\beta: Z^p(Y,B) \longrightarrow H^p(Y,B)$ and $\gamma: Z^p(Z,C) \longrightarrow H^p(Z,C)$. Then $f^*g^*(\gamma\varphi) = f^*\beta g^\#\varphi = \alpha f^\# g^\#\varphi = \alpha(gf)^\#\varphi = (gf)^*(\gamma\varphi)$.

2.20 <u>Theorem</u>: Let $f: (X,A,B) \longrightarrow (X',A',B')$ be continuous. Define $u: (X,A) \longrightarrow (X',A')$, $v: (X,B) \longrightarrow (X',B')$ and $w: (A,B) \longrightarrow (A',B')$ by $u(x) = v(x) = w(x) = f(x)$. Then the " ladder"

$$\begin{array}{ccccccccc}
\xrightarrow{\delta} & H^p(X',A') & \xrightarrow{j^*} & H^p(X',B') & \xrightarrow{i^*} & H^p(A',B') & \xrightarrow{\delta} \\
 & \downarrow u^* & & \downarrow v^* & & \downarrow w^* & \\
\xrightarrow[\delta]{} & H^p(X,A) & \xrightarrow[j^*]{} & H^p(X,B) & \xrightarrow[i^*]{} & H^p(A,B) & \xrightarrow[\delta]{}
\end{array}$$

is analytic (each rectangle of the ladder is analytic). (Axiom 3 c).

<u>Proof</u>: It is trivial to verify that $jv = uj$ and that $iv = wi$, hence $v^*j^* = j^*u^*$ and $v^*i^* = i^*w^*$. Now let $\alpha': Z^p(X',A') \longrightarrow H^p(X',A')$, $\beta': Z^p(X',B') \longrightarrow H^p(X',B')$, $\gamma': Z^p(A',B') \longrightarrow H^p(A',B')$, α, β, and γ as usual, and $t': (A',\square) \subset (X',\square)$. Then if $h \in H^p(A',B')$, $\varphi \in \gamma'^{-1}(h)$, and $\psi \in t'^{\#-1}(\varphi)$ with $\alpha'\bar{\delta}\psi = \delta h$. Thus $u^*\delta h = u^*\alpha'\bar{\delta}\psi = \alpha u^\#\bar{\delta}\psi = \alpha\bar{\delta}u^\#\psi$. But $t^\# u^\#\psi = w^\#\varphi$ implies $u^\#\psi \in t^{\#-1}(w^\#\varphi)$, hence $u^\#\psi \in t^{\#-1}\gamma^{-1}(\gamma w^\#\varphi)$ and so $\alpha\bar{\delta}u^\#\psi = \delta(\gamma w^\#\varphi) = \delta w^*h$, since $w^*h = \gamma w^\#\varphi$. Thus $\delta w^* = u^*\delta$. We now compute thus, $w^*i^*j^* = (w^*i^*)j^* = (i^*v^*)j^* = i^*(v^*j^*) = i^*(j^*u^*) = i^*j^*u^*$. Similarly $u^*\delta i^* = \delta i^*v^*$ and $v^*j^*\delta = j^*\delta w^*$.

2.21 <u>Corollary:</u> If $f: (X,A) \longrightarrow (X',A')$ is continuous and if $f(X) \subset A'$, then $f^*: H^p(X',A') \longrightarrow H^p(X,A)$ is the

zero function for each $p \geq 0$.

Proof: Recalling that $H^p(X,X) = \{0\}$ and noting that $f: (X,X,A) \longrightarrow (X',A',A')$ we use the previous theorem to assert that if $h \in H^p(X',A')$ then $f^*(h) = f^*j^*(h) = j^*f^*(h) \in j^*[H^p(X,X)] = \{0\}$.

2.22 Theorem: If X is connected then $H^0(X) \cong G$, and if X is a point space then $H^0(X) \cong G$ and $H^p(X) = \{0\}$ for $p \neq 0$. (Axiom 7 c).

Proof: If X is connected then each h in $Z^0(X)$ is a constant function, hence we may define $f: Z^0(X) \longrightarrow G$ by $f(h) = h(X)$. Clearly f is a monomorphism. If $g \in G$ define $h_g: X \longrightarrow G$ by $h_g(x) = g$ for each x in X. Then h_g is in $Z^0(X)$ and $f(h_g) = g$, hence f is an isomorphism and we have $G \cong Z^0(X) \cong H^0(X)$. Now assume $p \neq 0$ and $X = \{x\}$. If $\varphi \in Z^p(X)$ and $\varphi(x^{p+1}) = g$, then $\bar{\delta}\varphi(x^{p+2}) = \sum_{i=0}^{p+1}(-1)^i g = 0$, hence $\sum_{i=0}^{p}(-1)^i g = (-1)^p g$. Define $\psi: \{x^p\} \longrightarrow G$ by $\psi(x^p) = g$, then $\bar{\delta}\psi(x^{p+1}) = \sum_{i=0}^{p}(-1)^i g = (-1)^p g = (-1)^p \varphi(x^{p+1})$. Thus $\bar{\delta}\psi = \varphi$ or $\bar{\delta}(-\psi) = \varphi$, and we have $\varphi \in \bar{\delta}[C^{p-1}(X)] \subset B^p(X)$. Hence $Z^p(X) \subset B^p(X)$. Therefore equality holds and $H^p(X) = \{0\}$.

2.23 Lemma: If $f: (X,A) \subset (X,A)$, then $f^*: H^p(X,A) \subset H^p(X,A)$ for $p \geq 0$. (Axiom 1 c).

Proof: We have $f^*(\alpha\varphi) = \alpha f^{\#}(\varphi) = \alpha\varphi$.

2.24 Theorem: If $f: (X,A) \longrightarrow (Y,B)$ is a homeomorphism, then $f^*: H^p(Y,B) \longrightarrow H^p(X,A)$ is an isomorphism.

<u>Proof</u>: Let $g: (Y,B) \longrightarrow (X,A)$ be such that $gf(x) = x$ for each x in X and $fg(y) = y$ for each y in Y. Then $fg: (Y,B) \subset (Y,B)$, hence $g^*f^* = (fg)^*: H^p(Y,B) \subset H^p(Y,B)$ and $gf: (X,A) \subset (X,A)$, hence $f^*g^*: H^p(X,A) \subset H^p(X,A)$. Thus f^* is one to one and onto.

2.25 <u>Lemma</u>: Let $A \subset X$ and u be an open subset of the interior of A, then $C^p(X,A) \cap C^p(X,X-u) = C_0^p(X)$.

<u>Proof</u>: Clearly $C_0^p(X) \subset C^p(X,A) \cap C^p(X,X-u)$. If $\varphi \in C^p(X,A)$ then there exists a finite open cover $\mathcal{U}_1$ of A such that $\varphi = 0$ on $\mathcal{U}_1^{(p+1)} \cap A^{p+1}$, and $\varphi \in C^p(X,X-u)$ implies there exists a finite open cover $\mathcal{U}_2$ of $X-u \ni \varphi = 0$ on $\mathcal{U}_2^{(p+1)} \cap A^{p+1}$. Then if $\mathcal{U}$ denotes the collection of open sets obtained by intersecting the members of $\mathcal{U}_1$ with the interior of A, and the members of $\mathcal{U}_2$ with the complement of u closure, we have that $\mathcal{U}$ is a finite open cover of X such that $\varphi = 0$ on $\mathcal{U}^{(p+1)}$. Thus $\varphi \in C_0^p(X)$.

2.26 <u>Weak Excision Theorem</u>: If $k: (X-u,A-u) \subset (X,A)$ and if u is an open set contained in the interior of A, then $k^*: H^p(X,A) \longrightarrow H^p(X-u,A-u)$ is an isomorphism. (Axiom 6 c).

<u>Proof</u>: Let $\varphi \in Z^p(X-u,A-u)$ and define $\psi: X^{p+1} \longrightarrow G$ by the equations $\psi = \varphi$ on $(X-u)^{p+1}$; $\psi = 0$ elsewhere. Now $\varphi \in C^p(X-u,A-u)$ implies there exists a finite cover $\mathcal{U}$ of $A-u$ by sets open in $X-u$ such that $\varphi = 0$ on $\mathcal{U}^{(p+1)} \cap (A-u)^{p+1}$. Writing $\mathcal{U} = \{v_i \cap (A-u) | 1 \leq i \leq n\}$ where v_i is open in X and $\mathcal{V} = \{v_i | 1 \leq i \leq n\} \cup \{u\}$, we have that $\mathcal{V}$ is a finite open cover of A and $\psi = 0$ on $\mathcal{V}^{(p+1)} \cap$

A^{p+1}. Thus $\psi \in C^p(X,A)$ and $\bar{\delta}\psi \in C^{p+1}(X,A)$, with $\bar{\delta}k^{\#}\psi = \bar{\delta}\varphi \in C_0^{p+1}(X-u)$. Hence $k^{\#}\bar{\delta}\psi \in C_0^{p+1}(X-u)$ and so $\bar{\delta}\psi \in C^{p+1}(X,X-u)$. By the previous lemma then, $\bar{\delta}\psi \in C_0^{p+1}(X)$. Thus $\psi \in Z^p(X,A)$ and $k^{\#}$ is onto. To see that k^* is one to one we let $\varphi \in Z^p(X,A)$ such that $k^{\#}(\varphi) \in B^p(X-u,A-u)$. Then $k^{\#}(\varphi) = \varphi_1 + \bar{\delta}(\varphi_2)$, for some $\varphi_1 \in C_0^p(X-u)$ and $\varphi_2 \in C^{p-1}(X-u,A-u)$. Define ψ_2: $X^p \longrightarrow G$ by $\psi_2 = \varphi_2$ on $(X-u)^p$; $\psi_2 = 0$ elsewhere. Then as before $\psi_2 \in C^{p-1}(X,A)$ hence $\bar{\delta}(\psi_2) \in C^p(X,A)$ and so $\varphi - \bar{\delta}(\psi_2) \in C^p(X,A)$. Now $k^{\#}[\varphi - \bar{\delta}(\psi_2)] = k^{\#}\varphi - \bar{\delta}(k^{\#}\psi_2) = k^{\#}\varphi - \bar{\delta}(\varphi_2) = \varphi_1 \in C_0^p(X-u)$. Thus $\varphi - \bar{\delta}(\psi_2) \in C^p(X,X-u)$. Again by the previous lemma, we know there exists $\psi_1 \in C_0^p(X)$ such that $\varphi - \bar{\delta}(\psi_2) = \psi_1$. Hence $\varphi = \psi_1 + \bar{\delta}(\psi_2) \in C_0^p(X) + \bar{\delta}[C^{p-1}(X,A)] = B^p(X,A)$.

2.27 Theorem: Let f,g: $(X,A) \longrightarrow (Y,B)$ and let $\varphi \in C^p(Y)$. Let $\mathcal{V}$ be an open cover of Y such that $\bar{\delta}\varphi = 0$ on $\mathcal{V}^{(p+2)}$ and $\varphi = 0$ on $\mathcal{V}^{(p+1)} \cap B^{p+1}$. Finally let $\mathcal{U}$ be a finite open cover of X such that $u \in \mathcal{U}$ implies $f(u) \cup g(u) \cup v$ for some $v \in \mathcal{V}$. Then $f^{\#}\varphi$, $g^{\#}\varphi \in Z^p(X,A)$ and $f^{\#}\varphi - g^{\#}\varphi \in B^p(X,A)$.

Proof: $\mathcal{U}$ is a finite open cover of A and if $(x_0,\ldots,x_p) \in u^{p+1} \cap A^{p+1}$, where $u \in \mathcal{U}$, then $x_i \in u$ for $0 \leq i \leq p$. Hence there exists $v \in \mathcal{V}$ such that $f(u) \cup g(u) \subset v$. Then $f(x_i)$, $g(x_i) \in v \cap B$ for $0 \leq i \leq p$, hence $(f[x_0],\ldots,f[x_p])$, $(g[x_0],\ldots,g[x_p]) \in v^{p+1} \cap B^{p+1}$ and $f^{\#}\varphi(x_0,\ldots,x_p)$, $g^{\#}\varphi(x_0,\ldots,x_p) = 0$. Thus $f^{\#}\varphi$, $g^{\#}\varphi \in C^p(X,A)$. Also $[\bar{\delta}f^{\#}(\varphi)](x_0,\ldots,x_{p+1}) = [f^{\#}(\bar{\delta}\varphi)]$

$(x_0,\ldots,x_{p+1}) = \bar{\delta}\varphi(f[x_0],\ldots,f[x_{p+1}]) = 0$ and $[\bar{\delta}g^{\#}(\varphi)]$ $(x_0,\ldots,x_{p-1}) = 0$ for $(x_0,\ldots,x_{p+1}) \in u^{p+2}$, since $(f[x_0],\ldots,f[x_{p+1}])$ and $(g[x_0],\ldots,g[x_{p+1}])$ are in v^{p+2} and $\bar{\delta}\varphi(v^{p+2}) = 0$. Hence $\bar{\delta}g^{\#}\varphi$ and $\bar{\delta}f^{\#}\varphi \in C_0^{p+1}(X)$ and consequently $f^{\#}\varphi$, $g^{\#}\varphi \in Z^p(X,A)$. Now $\mathcal{U}$ is a finite open cover of X and if $(x_0,\ldots,x_p) \in u^{p+1}$ then there exists $v \in \mathcal{V}$ such that $f(x_i), g(x_i) \in v$, and consequently $[g(x_0),\ldots,g(x_i),$ $f(x_i),\ldots,f(x_p)] \in v^{p+2}$ for $0 \leq i \leq p$. Thus $\bar{\delta}\varphi[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_p)] = 0$ and so $\sum_{i=0}^{p}(-1)^i\bar{\delta}\varphi[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_p)] =$ $D\bar{\delta}\varphi(x_0,\ldots,x_p) = 0$. Hence $D\bar{\delta}\varphi = 0$ on $\mathcal{U}^{(p+1)}$ which implies $D\bar{\delta}\varphi \in C_0^p(X)$, and we **have** that for $p = 0$ $(f^{\#} - g^{\#})\varphi =$ $D\bar{\delta}\varphi \in C_0^p(X) \subset B^p(X,A)$. Also $\bar{\delta}[D\varphi] \in \bar{\delta}[C^{p-1}(X,A)]$ if $p \geq 1$, for if $(x_0,\ldots,x_p) \in u^p \cap A^p$ then $D\varphi(x_0,\ldots,x_{p-1}) =$ $\sum_{i=0}^{p-1}(-1)^i\varphi[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_{p-1})]$. But $f(x_i)$, $g(x_i) \in v \cap B$ for some $v \in \mathcal{V}$ and $0 \leq i \leq p-1$, hence $[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_{p-1})] \in v^{p+1} \cap B^{p+1}$ which implies $\varphi[g(x_0),\ldots,g(x_i), f(x_i),\ldots,f(x_{p-1})] = 0$ and therefore $D\varphi(u^p \cap A^p) = 0$. Thus $D\varphi \in C^{p-1}(X,A)$ and $f^{\#}\varphi - g^{\#}\varphi = \bar{\delta}D\varphi + D\bar{\delta}\varphi \in B^p(X,A)$.

The following Corollary is an extension of a fundamental lemma proved, at Wallace's suggestion, by Capel in [1].

2.28 <u>Corollary</u>: Let B be closed in the space Y and let $h \in H^p(Y,B)$. Then there exists such a finite open cover, $\mathcal{V}(h)$, of Y that; if $f,g\colon (X,A) \longrightarrow (Y,B)$ are maps such

that for each x in X, $f(x)$, $g(x) \in v(x)$ for some $v(x) \in \mathcal{V}(h)$, then $f^*(h) = g^*(h)$.

Proof: Let $h \in H^p(Y,B)$, then there exists $\varphi \in Z^p(Y,B)$ such that $\beta\varphi = h$. Since B is closed we apply lemma 2.5, part 4, to yield a finite open cover $\mathcal{V}$ of Y with $\delta\varphi = 0$ on $\mathcal{V}^{(p+2)}$ and $\varphi = 0$ on $B^{p+1} \cap \mathcal{V}^{(p+1)}$. Define $\mathcal{U} = \{f^{-1}(v) \cap g^{-1}(v) | v \in \mathcal{V}\}$. Then $\mathcal{U}$ is a finite open cover of X such that if $u \in \mathcal{U}$, there exists $v \in \mathcal{V}$ such that $f(u) \cup g(u) \subset v$. Thus $f^{\#}\varphi - g^{\#}\varphi \in B^p(X,A)$, and $f^*(h) = g^*(h)$.

2.29 Definition: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of a space X and let $C \subset X$.

1) $\mathcal{A}$ refines $\mathcal{B}$ ($\mathcal{A} < \mathcal{B}$) iff $A \in \mathcal{A}$ implies $A \subset B$ for some $B \in \mathcal{B}$.

2) $St(C, \mathcal{A}) = \cup\{A | A \in \mathcal{A}$ and $A \cap C \neq \square\}$.

3) $St(\mathcal{A}) = \{St(A, \mathcal{A}) | A \in \mathcal{A}\}$.

4) $\mathcal{A}$ star refines $\mathcal{B}$ iff $St(\mathcal{A}) < \mathcal{B}$.

We will need the following two results which are well known and will be stated without proof.

2.30 Theorem: A space X is normal iff for each finite open cover $\mathcal{U}$ of X there exists a finite open cover $\mathcal{V}$ such that $St(\mathcal{V}) < \mathcal{U}$.

2.31 Modification Lemma: If A is a subset of the space X, if $\mathcal{V}$ is an open cover of X, if $\mathcal{U}$ is an open cover of X such that $St(\mathcal{U}) < \mathcal{V}$ and if $P = St(A, \mathcal{U})$ then there is a function $f: (X,P) \longrightarrow (X,A)$ such that

i) $f(x) = x$ for $x \in A \cup (X-P)$

ii) If $u \in \mathcal{U}$ then there is a $v \in \mathcal{V}$ such that $u \cup f(u) \subset v$.

2.32 Definition: (X,A) is a normal pair iff X is a normal space and A is a closed subset of X.

2.33 Notation: If $i: (P,Q) \subset (R,S)$ and if $h \in H^p(R,S)$, then $h|(P,Q) = i^*(h)$. If $A \subset X$ we let A^o denote the interior of A, and A^* denote the closure of A.

2.34 Expansion Lemma: If (X,A) is a normal pair and if $h \in H^p(X,A)$, then there exists an open set $p \supset A$ and $h_0 \in H^p(X,P^*)$ such that $h_0|(X,A) = h$.

Proof: Let $h \in H^p(X,A)$, $\varphi \in Z^p(X,A)$ such that $\alpha(\varphi) = h$. Then there exists a finite open cover $\mathcal{V}$ of X such that $\varphi(v^{p+1} \cap A^{p+1}) = 0$ and $\bar{\delta}h(v^{p+2}) = 0$. By theorem 2.30 there exists a finite open cover $\mathcal{U}$ of X such that $St(\mathcal{U}) < \mathcal{V}$. By lemma 2.31 there exists $f: (X, St(A, \mathcal{U})) \longrightarrow (X,A)$ with $f(x) = x$ for $x \in A \cup (X-St(A,\mathcal{U}))$ and such that $u \in \mathcal{U}$ implies there exists $v \in \mathcal{V}$ such that $u \cup f(u) \subset v$. Let $P = P^o$ be such that $A \subset P \subset P^* \subset St(A,\mathcal{U})$, then $f^{\#}\varphi \in Z^p(X,P^*)$, for if $(x_0,\ldots,x_p) \in u^{p+1} \cap P^{*p+1}$ then $f(x_i) \in f(u) \cap A$ and $\exists\ v \in \mathcal{V}$ such that $u \cup f(u) \subset v$. Hence $f^{\#}\varphi(x_0,\ldots,x_p) = \varphi(f[x_0],\ldots,f[x_p]) \in \varphi(v^{p+1} \cap A^{p+1}) = 0$ and so $f^{\#}\varphi \in C^p(X,P^*)$. Similarly if $(x_0,\ldots,x_{p+1}) \in u^{p+2}$, then $(f[x_0],\ldots,f[x_{p+1}]) \in v^{p+2}$ for some $v \in \mathcal{V}$, hence $\bar{\delta}f\ \varphi(x_0,\ldots,x_{p+1}) = f\ \bar{\delta}\varphi(x_0,\ldots,x_{p+1}) = \bar{\delta}\varphi(f[x_0],\ldots,f[x_{p+1}])$

$\in \bar{\delta}\varphi(v^{p+2}) = 0$. Thus $f^{\#}\varphi \in \bar{\delta}^{-1}[C_0^{p+1}(X)]$ and so $f^{\#}\varphi \in Z^p(X,P^*)$. Now if $i: (X,A) \subset (X,A)$, then we have $i,f: (X,A) \longrightarrow (X,A)$ and φ, $\mathcal{U}$, $\mathcal{V}$ satisfy the conditions of theorem 2.27, hence $i^{\#}\varphi - f^{\#}\varphi \in B^p(X,A)$. Consequently if we let $\varrho: Z^p(X,P^*) \longrightarrow H^p(X,P^*)$ denote the natural homomorphism, then taking $h_0 = \varrho[f^{\#}\varphi]$ we have $h_0|(X,A) = h$.

2.35 Theorem: Let (X,A) be a normal pair and let $A \subset M^o = M \subset X$. Then, if $h \in H^p(X,M^*)$ and if $h|(X,A) = 0$, there exists an open set N such that $A \subset N \subset N^* \subset M$ and $h|(X,N^*) = 0$.

Proof: $i: (X,A) \subset (X,M^*)$. Let $h \in H^p(X,M^*)$, $\varphi \in Z^p(X,M^*)$ with $\varrho\varphi = h$. Then $i^*(h) = 0$ implies $i^{\#}\varphi \in B^p(X,A)$, hence $\varphi = \varphi_1 + \bar{\delta}(\varphi_2)$ where $\varphi_1 \in C^p(X)$, $\varphi_2 \in C^{p-1}(X,A)$ for $p \geq 1$. Thus there exists $\mathcal{V}_1$, a finite open cover of X such that $\varphi_2 = 0$ on $\mathcal{V}_1^{(p)} \cap A^p$, $\mathcal{V}_2$ a finite open cover of X such that $\varphi_1 = 0$ on $\mathcal{V}_2^{(p+1)}$, and $\mathcal{V}_3$ a finite open cover of X such that $\varphi = 0$ on $\mathcal{V}_3^{(p+1)} \cap M^{*p+1}$. Let $\mathcal{V} = \{v_1 \cap v_2 \cap v_3 | v_i \in \mathcal{V}_i,\ i = 1,2,3,\}$, then $\mathcal{V}$ is a finite open cover of X satisfying the same conditions as $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ separately. Let $\mathcal{U}$ be a finite open cover of X such that $St(\mathcal{U}) < \mathcal{V}$ and let $N = N^o$ be such that $A \subset N \subset N^* \subset [St(A,\mathcal{U}) \cap M]$. Then there exists $f: (X, St(A,\mathcal{U})) \longrightarrow (X,A)$ with $f(x) = x$ for $x \in A \cup [X - St(A,\mathcal{U})]$ and $u \in \mathcal{U}$ implies there exists $v \in \mathcal{V}$ such that $u \cup f(u) \subset v$. Now as before $i,f: (X,N^*) \longrightarrow (X,N^*)$ and φ, $\mathcal{U}$, $\mathcal{V}$, satisfy the conditions of theorem 2.27,

hence $i^{\#}\varphi - f^{\#}\varphi \in B^p(X,N^*)$. But $f^{\#}\varphi = f^{\#}\varphi_1 + f^{\#}\bar{\delta}(\varphi_2) = f^{\#}\varphi_1 + \bar{\delta}(f^{\#}\varphi_2)$. By a proof similar to that of the expansion lemma $f^{\#}\varphi_2 \in C^{p-1}(X,N^*)$, and if $(x_0,\ldots,x_p) \in u^{p+1}$ then there exists v such that $u \cup f(u) \subset v$ so that $(f[x_0],\ldots,f[x_p]) \in v^{p+1}$. Thus $f^{\#}\varphi_1(x_0,\ldots,x_p) = \varphi_1(f[x_0],\ldots,f[x_p]) = 0$ and consequently $f^{\#}\varphi_1 \in C_0^p(X)$. But then $f^{\#}\varphi \in B^p(X,N^*)$ so that $\varphi - f^{\#}\varphi \in B^p(X,N^*)$ implies $\varphi \in B^p(X,N^*)$. Finally $f\varphi = h$ and we have $h|(X,N^*) = 0$.

2.36 Notation: Through the Map Excision Theorem let (X,A) and (Y,B) be normal pairs and let $f: (X,A) \longrightarrow (Y,B)$ be a closed map such that f takes $X - A$ topologically onto $Y - B$, that is, $g: (X-A) \longrightarrow (Y-B)$ where $g(x) = f(x)$ is a homeomorphism.

2.37 Lemma: If $M_1, M_2 \supset A$, then $M_i = f^{-1}f(M_i)$ for $i = 1, 2$, and $f(M_1 - M_2) = f(M_1) - f(M_2) = [B \cup f(M_1)] - [B \cup f(M_2)]$.

2) If M is an open set containing A, then $B \cup f(M)$ is open.

3) If $B \subset N = B \cup ff^{-1}(N)$

4) If N is an open set containing B then $f^{-1}(N^*) = [f^{-1}(N)]^*$.

2.38 Lemma: Let M be an open set containing A and define $f_0: (X,M^*) \longrightarrow (Y,B \cup f(M^*))$ by $f_0(x) = f(x)$ for each $x \in X$. Then $f_0^*: H\ (Y,B \cup f(M^*)) \cong H\ (X,M^*)$.

Proof: Choose $N = N^o$ such that $A \subset N \subset N^* \subset M = M^o$.

Then if we let $f_1: (X-N, M^*-N) \longrightarrow (Y-[B \cup f(N)],$ $[B \cup f(M^*)] - [B \cup f(N)])$ be defined by $f_1(x) = f(x)$, we have that f_1 is an onto homeomorphism. For $f(X-N) =$ $[B \cup f(X)] - [B \cup f(N)] = Y-[B \cup f(N)]$ and $X - A$ contains $X - N$. Similarly $f(M^*-N) = [B \cup f(M^*)] - [B \cup f(N)]$. Now $f(N^*) = [f(N^*)]^*$ implies $[f(N)]^* \subset f(N^*) \subset f(M)$, thus $[B \cup f(N)]^* = B^* \cup [f(N)]^* = B \cup [f(N)]^* \subset B \cup f(N^*) \subset$ $B \cup f(M) \subset B \cup f(M^*)$ and hence $B \cup f(M)$ open implies $[B \cup f(N)]^* \subset [B \cup f(M^*)]^o$. Now let $i: (Y-[B \cup f(N)],$ $[B \cup f(M^*)] - [B \cup f(N)]) \subset (Y, B \cup f(M^*))$; $k: (X-N, M^*-N) \subset$ (X,M^*). By theorems 2.24 and 2.26 we have that f_1^*, i^*, and k^* are isomorphisms, hence $f_0^* = k^{*-1} f^* i^*$ is an isomorphism.

2.39 <u>Map Excision Theorem</u>: $f^*: H^p(Y,B) \cong H^p(X,A)$.

<u>Proof</u>: Let $h \in H^p(X,A)$, then there exists $M = M^o \supset A$ and $h_o \in H^p(X,M^*)$ such that $i^*(h_o) = h$; $i: (X,A) \subset (X,M^*)$. Let $h_1 = j^* f_0^{*-1}(h_o) \in H^p(Y,B)$, $j: (Y,B) \subset (Y, B \cup f(M^*))$; then $f^*(h_1) = f^* j^* f_0^{*-1}(h_o) = i^* f_0^* f_0^{*-1}(h_o) = i^*(h_o) = h$. Thus f^* is an epimorphism. Let $h \in K(f^*)$ then there exists $N = N^o \supset B$ and $h_o \in H^p(X,N^*)$ such that $l^*(h_o) = h$, $l: (Y,B) \subset$ (Y,N^*). Thus if $f_1: (X, f^{-1}(N)) \longrightarrow (Y,N^*)$ is defined by $f_1(x) = f(x)$, then $f_1^*(h_o) \in H^p(X, f^{-1}(N))$ is such that $f_1^*(h_o) | (X,A) = 0$. Hence there exists $M = M^o$ such that $A \subset M \subset M^* \subset f^{-1}(N)$ and $k^* f_1^*(h_o) = f_1^*(h_o) | (X,M^*) = 0$. Let $i_1: (Y, B \cup f(M^*)) \subset (Y,N^*)$, $j: (Y,B) \subset (Y, B \cup f(M^*))$, then $j^* i^* = l^*$ and $h = j^* i_1^*(h_o) = j^* f_0^{*-1} k^* f_1^*(h_o) = j^* f_0^{*-1}(0) = 0$. Thus f^* is a monomorphism.

2.40 Full Excision Theorem: If X is a normal space and if $X = A \cup B$ where A and B are closed subsets of X and if $f\colon (A, A \cap B) \subset (X, B)$, then $f^*\colon H^p(X,B) \cong H^p(A, A \cap B)$.

Proof: We have $A - (A \cap B) = A - B = (A \cup B) - B = X - B$, and $A = A^*$ implies that sets closed in A are closed in X, so that f is closed. Thus the hypotheses of the Map Excision theorem are satisfied and the conclusion follows.

2.41 Lemma: In the diagram of groups and homomorphisms, if the diagram is analytic and $h = gf$ is an isomorphism, then f is a monomorphism, g is an epimorphism, $Q = I(f) + K(g)$, and g takes $I(f)$ isomorphically onto R.

$$\begin{tikzcd} & Q \arrow[dr, "g"] & \\ P \arrow[ur, "f"] \arrow[rr, "h"'] & & R \end{tikzcd}$$

2.42 Lemma: In the diagram of groups and homomorphisms, if each triangle is analytic, $I(f_1) = K(g_1)$, $I(f_2) = K(g_2)$, and h_1, h_2 are isomorphisms; then:

i) $f\colon P_1 \times P_2 \cong Q$ where $f(p_1, p_2) = f_1(p_1) + f_2(p_2)$

ii) $g_1 \times g_2\colon Q \cong R_1 \times R_2$

iii) If $q \in Q$, then $q = f_2h_2^{-1}g_2(q) + f_1h_1^{-1}g_1(q)$

$$\begin{tikzcd} R & & R \\ & Q \arrow[ul, "g_1"'] \arrow[ur, "g_2"] & \\ P \arrow[uu, "h_1"] \arrow[ur, "f_1"'] & & P \arrow[uu, "h_2"'] \arrow[ul, "f_2"] \end{tikzcd}$$

2.43 Corollary: If $X = A \cup B$, X is normal and A and B are

closed, then the appropriate inclusion maps induce isomorphisms; $H^p(X,A) \times H^p(X,B) \cong H^p(X,A \cap B) \cong H^p(B,A \cap B) \times H^p(A,A \cap B)$.

Proof: Consider the diagram:

$$\begin{tikzcd}
H^p(B,A \cap B) & & H^p(A,A \cap B) \\
& H^p(X,A \cap B) \arrow[ul, "i_2^*"'] \arrow[ur, "j_2^*"] & \\
H^p(X,A) \arrow[uu, "i^*"] \arrow[ur, "i_1^*"'] & & H^p(X,B) \arrow[uu, "j^*"'] \arrow[ul, "j_1^*"]
\end{tikzcd}$$

we have $I(i_1^*) = K(j_2^*)$ and $I(j_1^*) = K(i_2^*)$, analyticity, and by the Full Excision Theorem i^*, j^* are isomorphisms. Thus $i_2^* \times j_2^*$ and φ, defined by $\varphi(h_1, h_2) = i_1^*(h_1) + j_1^*(h_2)$ are isomorphisms by the previous lemma.

2.44 Map Addition Theorem: Let $X = X_1 \cup X_2$ be normal with X_1 and X_2 closed and $A = X_1 \cap X_2$. If f_1, f_2, f: $(X,A) \longrightarrow (Y,B)$ are continuous and if $f_i(x) = f(x)$ for each $x \in X_i$, $f(X_i) \subset B, i = 1,2$; then $f^* = f_1^* + f_2^*$.

Proof: Consider the following diagram, where the homomorphisms not induced by the mappings mentioned in the hypotheses are induced by the corresponding inclusion maps.

$$\begin{tikzcd}
H^p(X_1,X_1 \cap X_2) & & H^p(X_2,X_1 \cap X_2) \\
& H^p(X,X_1 \cap X_2) \arrow[ul, "i_1^*"'] \arrow[ur, "j_2^*"] & \\
H^p(X,X_2) \arrow[uu, "i^*"] \arrow[ur, "i_1^*"'] & & H^p(X,X_1) \arrow[uu, "j^*"'] \arrow[ul, "j_1^*"] \\
& H^p(Y,B) \arrow[uu] \arrow[ul, "f_2^*"] \arrow[ur, "f_1^*"'] &
\end{tikzcd}$$

We use lemma 2.42 to write $f^*(h) = i_1^* i^{*-1} i_2^* f^*(h) +$

$j_1^* j^{*-1} j_2^* f^*(h)$. By analyticity then $f^*(h) = i_1^* f_2^*(h) + j_1^* f_1^*(h) = f_2^*(h) + f_1^*(h)$ and so $f^* = f_1^* + f_2^*$.

2.45 <u>Reduction Theorem</u>: Let (X, X_0) be a normal pair and let A be closed in X. If $h \in H^p(X, X_0)$ and if $h|(A, A \cap X_0) = 0$, then there exists an open set $M \supset A \cup X_0$ such that $h|(M^*, X_0) = 0$. Hence there exists an open set N about A such that $h|(N^*, N^* \cap X_0) = 0$.

<u>Proof</u>: Consider the following diagram:

$$\begin{tikzcd}
H^p(M^*, X_0) \arrow[r, "f_1^*"] & H^p(M^*, M^* \cap X_0) & \\
H^p(X, X_0) \arrow[u, "f^*"] \arrow[ur, "f_2^*"] \arrow[d, "k^*"] \arrow[dr, "i_0^*"] & H^p(X, A \cup X_0) \arrow[l, "i_1^*"] & H^p(X, M^*) \arrow[l, "i_2^*"] \arrow[ll, bend right, "g^* = i_1^* i_2^*"'] \\
H^p(A, A \cap X_0) & H^p(A \cup X_0, X_0) \arrow[l, "j^*"] &
\end{tikzcd}$$

where all the homomorphisms are induced by the corresponding inclusion maps. Let $h \in H^p(X, X_0)$ and $k^*(h) = 0$, then $i^*(h) = 0$ since j^* is an isomorphism by the Full Excision theorem and $k_1^* = j^* i_0^*$; hence $h \in K(i_0^*)$. Also, we get $I(i_1^*) = K(i_0^*)$ from theorem 2.16 applied to the triple $(X, A \cup X_0, X_0)$, and so there exists $h_o \in H^p(X, A \cup X_0)$ such that $i_1^*(h_o) = h$. By the Expansion Lemma there exists an open set $M \supset (A \cup X_0)$ and $h_o' \in H^p(X, M^*)$ such that $i_2^*(h_o') = h_o$. Applying theorem 2.16 to the triple (X, M^*, X_0) we have $K(f^*) = I(g^*)$. Thus $h|(M^*, X_0) = g^*(h_o') = h$ implies $f^*(h) = 0$. Now let $M = N$ for the second part of the theorem.

2.46 Extension Theorem: Let (X,X_0) be a normal pair and let A be closed in X. If $h \in H^p(A, A \cap X_0)$ then there exists an open set $M \supset (A \cup X_0)$ and $h_0' \in H^p(M^*, X_0)$ such that $h_0' | (A, A \cap X_0) = h$. Thus there is an open set N about A and $h_0 \in H^p(N^*, N^* \cap X_0)$ such that $h_0 | (A, A \cap X_0) = h$.

Proof: Consider the following diagram:

$$\begin{tikzcd}
H^p(X,X_0) \arrow[r, "k^*"] \arrow[dr, "j^*"'] & H^p(A \cup X_0, X_0) \arrow[r, "\delta"] & H^{p+1}(X, A \cup X_0) \arrow[r] & H^{p+1}(X,X_0) \\
& H^p(M^*, X_0) \arrow[u, "i^*"] \arrow[r, "\delta"] & H^{p+1}(X,M^*) \arrow[u, "i_1^*"] \arrow[ur] &
\end{tikzcd}$$

Let $h \in H^p(A \cup X_0, X_0)$. Then by the Expansion Lemma, there exists $M = M^o \supset (A \cup X_0)$ and $h_1 \in H^{p+1}(X, M^*)$ such that $h_1 | (X, A \cup X_0) = \delta(h)$, thus $h_1 | (X,X_0) = \delta(h) | (X,X_0) = 0$ by exactness. Hence there exists $h_2 \in H^p(M^*, X_0)$ such that $\delta(h_2) = h_1$. Then $\delta[h - i^*(h_2)] = \delta(h) - \delta i^*(h_2) = \delta(h) - i^*\delta(h_2) = 0$ implies $i^*(h_2) \in K(\delta) = I(k^*)$. Thus there exists $h_3 \in H^p(X,X_0)$ such that $k^*(h_3) = h - i^*(h_2)$. Define $h_0' = h_2 + j^*(h_3) \in H^p(M^*, X_0)$, then $h_0' | (A \cup X_0, X_0) = i^*[h_2 + j^*(h_3)] = i^*(h_2) + i^*j^*(h_3) = i^*(h_2) + k^*(h_3) = i^*(h_2) + h - i^*(h_2) = h$. Now $f^*: H^p(A \cup X_0, X_0) \cong H^p(A, A \cap X_0)$ hence we let $h \in H^p(A, A \cap X_0)$ then there exists $M = M^o \supset (A \cup X_0)$ and $h_0' \in H^p(M^*, X_0)$ such that $i^*(h_0') = f^{*-1}(h)$. Therefore $f^* i^*(h_0') = h$, or equivalently $h_0' | (A, A \cap X_0) = h$. For the second part of the theorem let $M = N$.

2.47 <u>Lemma</u>: In the diagram of groups and homomorphisms:

$$\begin{tikzcd}
 & G_o \arrow[dl, "t_1"'] \arrow[dr, "t_2"] \arrow[dd, "j_o"] & \\
H_1 & & H_2 \\
 & G \arrow[ul, "i_1"] \arrow[ur, "i_2"'] \arrow[dd] & \\
G_2 \arrow[uu, "k_1"] \arrow[ur, "j_2"'] \arrow[dr, "h_1"'] & & G_1 \arrow[uu, "k_2"'] \arrow[ul, "j_1"] \arrow[dl, "h_2"] \\
 & H_o &
\end{tikzcd}$$

if each triangle is analytic, $I(j_\alpha) = K(i_\alpha)$ for $\alpha = 1,2$, $i_o j_o = 0$, and k_1, k_2 are isomorphisms, then $h_1 k_1^{-1} t_1 = -h_2 k_2^{-1} t_2$

2.48 <u>Theorem</u>: Let X be a space and let $B \subset A \subset X_1 \cap X_2 \subset X_1 \cup X_2 = X$. Let $(j^*_\alpha, i^*_\alpha, \delta_\alpha)$, $(\tilde{j}^*_\alpha, \tilde{i}^*_\alpha, \tilde{\delta}_\alpha)$, $(\bar{j}^*_\alpha, \bar{i}^*_\alpha, \bar{\delta}_\alpha)$, and (j^*, i^*, δ) be the homomorphisms in the exact sequences for the triples (X, X_α, B), (X, X_α, A), (X_α, A, B), and (X, A, B) respectively for $\alpha = 1,2$. If $k_1: (X_2, A) \subset (X, X_1)$ and $k_2: (X_1, A) \subset (X, X_2)$ induce isomorphisms for each $p \geq 0$, then there exists an exact sequence:

$$\cdots \xrightarrow{I^*} H^{p-1}(A,B) \xrightarrow{\Delta} H^p(X,B) \xrightarrow{J^*} H^p(X_1,B) \times H^p(X_2,B) \xrightarrow{I^*} H^p(A,B) \xrightarrow{\Delta} \cdots$$

where $J^* = i_1^* \times i_2^*$, $I^* = \bar{i}_1^* - \bar{i}_2^*$, and $\Delta = j_2^* k_2^{*-1} \bar{\delta}_1 = -j_1^* k_1^{*-1} \bar{\delta}_2$.

<u>Proof</u>: Consider the following diagram:

$$
\begin{tikzcd}
 & & H^{p-1}(A,B) \arrow[dll, "\bar\delta_2"'] \arrow[drr, "\bar\delta_1"] \arrow[dd] & & \\
H^p(X_2,A) & & & & H^p(X_1,A) \\
 & & H^p(X,A) \arrow[ull, "\tilde{i}_2^*"'] \arrow[urr, "\tilde{i}_1^*"] \arrow[dd] & & \\
H^p(X,X_1) \arrow[uu, "k_1^*"] \arrow[urr, "\tilde{j}_1^*"] \arrow[drr, "j_1^*"] & & & & H^p(X,X_2) \arrow[uu, "k_2^*"'] \arrow[ull, "\tilde{j}_2^*"'] \arrow[dll, "j_2^*"'] \\
 & & H^p(X,B) \arrow[dll, "i_1^*"'] \arrow[drr, "i_2^*"] \arrow[dd] & & \\
H^p(X_1,B) \arrow[drr, "\bar{i}_1^*"] & & & & H^p(X_2,B) \arrow[dll, "\bar{i}_2^*"'] \\
 & & H^p(A,B) & &
\end{tikzcd}
$$

That the conditions of lemma 2.47 are satisfied follows from theorem 2.20, hence $j_1^* k_1^{*-1}\delta_2 = -j_2^* k_2^{*-1}\delta_1$. The proof proceeds with seven parts.

1) Let $h \in Z^0(X,B)$ with $(i_1^\#(h), i_2^\#(h)) = 0$ in $Z^0(X_1,B) \times Z^0(X_2,B)$, then $i_1^\#(h) = 0$ and $i_2^\#(h) = 0$. Hence if x is in X_1 then $[i_1^\#(h)](x) = h[i_1(x)] = h(x) = 0$, similarly x in X_2 implies $h(x) = 0$, therefore $h(x) = 0$ for all x in X and we have $h = 0$. Thus J^* is a monomorphism on $H^0(X,B)$.

2) $I(\Delta) \subset K(J^*)$. If $h \in I(\Delta)$ then there exists $h' \in H^p(A,B)$ such that $h = j_1^* k_1^{*-1}\delta_2(h') = -j_2^* k_2^{*-1}\bar\delta_1(h')$. Hence $i_1^*(h) = i_1^* j_1^*[k_1^{*-1}\bar\delta_2(h')] = 0$, similarly $i_2^*(h) = 0$. Thus $J^*(h) = (i_1^*(h), i_2^*(h)) = (0,0)$ and $h \in K(J^*)$.

3) $K(J^*) \subset I(\Delta)$. Suppose $J^*(h) = (0,0)$, then $i_1^*(h) = i_2^*(h) = 0$ and $h \in K(i_1^*) = I(j_1^*)$. Hence there exists $h_1 \in H^p(X,X_1)$ such that $j_1^*(h_1) = h$. Now $\bar{j}_2^* k_1^* = i_2^* j_1^*$ and

$\bar{j}_2^* k_1^*(h_1) = i_2^* j_1^*(h_1) = i_2^* j_1^*(h_1) = i_2^*(h) = 0$. Hence $k_1^*(h_1) \in K(\bar{j}_2^*) = I(\bar{\delta}_2)$ and $h_1' \in H^{p-1}(A,B)$ such that $\bar{\delta}_2(h_1') = k_1^*(h_1)$, which implies $k_1^{*-1}\bar{\delta}_2(h_1') = h_1$ and so $j_1^* k_1^{*-1}\bar{\delta}_2(h_1') = j_1^*(h_1) = h$.

4) $I(J^*) \subset K(I^*)$. If $(h_1,h_2) \in I(J^*)$ then there exists $h \in H^p(X,B)$ such that $J^*(h) = (h_1,h_2) = (i_1^*(h), i_2^*(h))$. Therefore $I^*(h_1,h_2) = I^*(i_1^*(h), i_2^*(h)) = \bar{i}_1^* i_1^*(h) - \bar{i}_2^* i_2^*(h) = i^*(h) - i^*(h) = 0$, and $(h_1,h_2) \in K(I^*)$.

5) $K(I^*) \subset I(J^*)$. We note that $\bar{\delta}_1 \bar{i}_2^* = k_2^* \delta_2$ and let $(h_1,h_2) \in K(I^*)$. Then $\bar{i}_1^*(h_1) - \bar{i}_2^*(h_2) = 0$, hence $k_2^*\delta_2(h_2) = \bar{\delta}_1 \bar{i}_1^*(h_1) = 0$. Thus $h_2 \in K(\delta_2) = I(i_2^*)$ and there exists $h_2' \in H^p(X,B)$ such that $i_2^*(h_2') = h_2$. Similarly $h_1 \in I(i_1^*)$ and hence there exists $h_1' \in H^p(X,B)$ such that $i_1^*(h_1') = h_1$. Now $i^*(h_1'-h_2') = i^*(h_1') - i^*(h_2') = \bar{i}_1^* i_1^*(h_1') - \bar{i}_2^* i_2^*(h_2') = \bar{i}_1^*(h_1) - \bar{i}_2^*(h_2) = 0$, hence there exists $h_o \in H^p(X,A)$ such that $j^*(h_o) = h_1' - h_2'$. But $H^{p-1}(X,A) = I(\tilde{j}_1^*) + I(\tilde{j}_2^*)$ implies $h_o = \tilde{j}_1^*(\varphi_1) + \tilde{j}_2^*(\varphi_2)$ where $\varphi_1 \in H^p(X,X_1)$ and $\varphi_2 \in H^p(X,X_2)$. Thus $j^*(h_o) = j^*\tilde{j}_1^*(\varphi_1) + j^*\tilde{j}_2^*(\varphi_2) = j_1^*(\varphi_1) + j_2^*(\varphi_2) = h_1' - h_2'$, which implies $h_1' - j_1^*(\varphi_1) = h_2' + j_2^*(\varphi_2)$. Let $h = h_1' - j_1^*(\varphi_1) = h_2' + j_2^*(\varphi_2) \in H^p(X,B)$. Then $J^*(h) = (i_1^*(h), i_2^*(h)) = (i_1^*[h_1' - j_1^*(\varphi_1)], i_2^*[h_2' + j_2^*(\varphi_2)]) = (i_1^*(h_1') - i_1^* j_1^*(\varphi_1), i_2^*(h_2') - i_2^* j_2^*(\varphi_2)) = (i_1^*(h_1'), i_2^*(h_2')) = (h_1,h_2)$. Therefore $(h_1,h_2) \in I(J^*)$.

6) $I(I^*) \subset K(\Delta)$. If $(h_1,h_2) \in H^p(X_1,B) \times H^p(X_2,B)$ then $I^*(h_1,h_2) = \bar{i}_1^*(h_1) - \bar{i}_2^*(h_2)$ and $\bar{\delta}_2[\bar{i}_1^*(h_1) - \bar{i}_2^*(h_2)] = \bar{\delta}_2 \bar{i}_1^*(h_1) = k_1^*\delta_1(h_1)$. Thus $j_1^* k_1^{*-1}\bar{\delta}_2[\bar{i}_1^*(h_1) - \bar{i}_2^*(h_2)] = j_1^* k_1^{*-1} k_1^* \delta_1(h_1) = j_1^*\delta_1(h_1) = 0$.

7) $K(\Delta) \subset I(I^*)$. Let $h \in K(\Delta)$, then $j_1^* k_1^{*-1} \bar{\delta}_2(h) = 0$, hence $k_1^{*-1}\bar{\delta}_2(h) \in K(j_1^*) = I(\delta_1)$. Thus there exists $h_1 \in H^p(X,B)$ such that $\delta_1(h_1) = k_1^{*-1}\bar{\delta}_2(h)$ which implies that $k_1^*\delta_1(h_1) = \bar{\delta}_2(h)$ and so $\bar{\delta}_2\bar{i}_1^*(h_1) = \bar{\delta}_2(h)$ or $\bar{\delta}_2[\bar{i}_1^*(h_1) - h] = 0$. Since then $\bar{i}_1^*(h_1) - h \in K(\bar{\delta}_2) = I(\bar{i}_2^*)$ there exists $h_2 \in H^p(X_2,B)$ such that $\bar{i}_2^*(h_2) = \bar{i}_1^*(h_1) - h$ and $h = \bar{i}_1^*(h_1) - \bar{i}_2^*(h_2) = I^*(h_1,h_2)$. Consequently, $h \in I(I^*)$.

2.49 Absolute Mayer-Vietoris Sequence: Let X be normal and let X_1, X_2 and X_0 be closed subsets of X and let $X = X_1 \cup X_2$. Then there exists an exact sequence;

$$\cdots \xrightarrow{I_0} H^{p-1}(X_1 \cap X_2,\ X_1 \cap X_2 \cap X_0) \xrightarrow{\Delta_0} H^p(X,X_0)$$

$$\xrightarrow{J_0} H^p(X_1,X_1 \cap X_0) \times H^p(X_2,X_2 \cap X_0) \xrightarrow{I_0} H^p(X_1 \cap X_2, X_1 \cap X_2 \cap X_0)$$

$$\xrightarrow{\Delta_0} \cdots$$

where letting $t_\alpha: (X_\alpha, X_\alpha \cap X_0) \subset (X,X_0)$, $s_\alpha: (X_1 \cap X_2,\ X_1 \cap X_2 \cap X_0) \subset (X_\alpha, X_\alpha \cap X_0)$ for $\alpha = 1,2$, we have $J_0 = t_1^* \times t_2^*$ and $I_0 = s_1^* - s_2^*$.

Proof: Write $X_1' = X_1 \cup X_0$, $X_2' = X_2 \cup X_0$, $A = X_1' \cap X_2' = (X_1 \cap X_2) \cup X_0$ and $B = X_0$. Then $k_1: (X_2 \cup X_0, (X_1 \cap X_2) \cup X_0) \subset (X, X_1 \cup X_0)$ and $k_2: (X_2 \cup X_0, (X_1 \cap X_2) \cup X_0) \subset (X, X_2 \cup X_0)$ are isomorphisms by the Full Excision Theorem. Hence we combine theorems 2.20 and 2.48 to write the analytic "ladder" below, with exact upper "leg".

$$\begin{tikzcd}
\arrow[r, "\Delta"] & H^p(X,X_0) \arrow[r, "J^*"] \arrow[d, "w_3^*"] & H^p(X_1 \cup X_0,X_0) \times H^p(X_2 \cup X_0,X_0) \arrow[d, "w_1^*"] \arrow[d, "w^*"] \arrow[d, "w_2^*"] \\
\arrow[r, "\Delta_o"] & H^p(X,X_0) \arrow[r, "J_0"] & H^p(X_1,X_1 \cap X_0) \times H^p(X_2,X_2 \cap X_0)
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "I^*"] & H^p([X_1 \cap X_2] \cup X_0,X_0) \arrow[r] \arrow[d, "w_4^*"] & {} \\
\arrow[r, "I_0"] & H^p(X_1 \cap X_2,X_1 \cap X_2 \cap X_0) &
\end{tikzcd}$$

The homomorphisms $w_i^*, i = 1,2,3,4$ are understood to be induced by the appropriate inclusion maps and $w^* = w_1^* \times w_2^*$. By the Full Excision Theorem w_1^*, w_2^*, and w_4^* are isomorphisms, while w_3^* is the identity isomorphism. We define $J_0 = w^* J^* w_3^{*-1}$, $I_0 = w_4^* I^* w^{*-1}$, $\Delta_o = w_3^* \Delta w_4^{*-1}$. Now $I_0 J_0 = w_4^* I^* w^{*-1} w^* J^* w_3^{*-1} = w_4^* I^* J^* w_3^{*-1} = 0$, hence $I(J_0) \subset K(I_0)$. Conversely if $I_0(\varphi) = 0$, then $w_4^* I^* w^{*-1}(\varphi) = 0$ which implies $I^* w^{*-1}(\varphi) = 0$ and $w^{*-1}(\varphi) \in K(I^*)$. Thus $w^{*-1}(\varphi) = J^*(\theta)$, hence $\varphi = w^* J^*(\theta) = J_0 w_3^*(\theta)$ and $\varphi \in I(J_0)$. Consequently $K(I_0) \subset I(J_0)$. Similar computations show $I(I_0) = K(\Delta_o)$ and $I(\Delta_o) = K(J_0)$ and that $I_0 = s_1^* - s_2^*$, $J_0 = t_1^* \times t_2^*$.

2.50 <u>Theorem</u>: If X is connected, normal and T_1, and if $H^1(X) = \{0\}$, then X is unicoherent. ($G \neq \{0\}$)

<u>Proof</u>: Let A,B be closed connected subsets of X with $X = A \cup B$. Using the Mayer-Vietoris Sequence with $X_1 = A, X_2 = B, X_0 = \{x\}$ for fixed $x \in A \cap B$ we have the following diagram:

$$H^0(X,x) \xrightarrow{J_o} \begin{matrix} H^0(A,x) \\ \times \\ H^0(B,x) \end{matrix} \xrightarrow{I_o} H^0(A \cap B,x) \xrightarrow{\Delta_o} H^1(X,x) = H^1(X)$$

Since $H^0(X,x) = \{0\}$ by theorem 2.10, and $H^1(X,x) \cong H^1(X) = \{0\}$ by assumption and by theorem 2.16 and corollary 2.18, we have that I_0 is an isomorphism. Applying theorem 2.10 again yields the desired result.

CHAPTER III

A GROUP ASSIGNMENT FOR A SPECIAL CLASS OF SPACES

We let X be a space with the property that the intersection of two arbitrary open connected sets is the union of open connected sets. (e.g. any locally connected space, or the circle after replacing a proper subarc by the closure of the sin (x^{-1}) curve). Then if G is a fixed abelian group we will define, for each positive integer p, an abelian group $H^p(X)$ in much the same way as in Chapter II; with the notable distinction that now the open covers of interest will consist of connected sets. Initially it was hoped that this new group assignment would be such that spaces which were not locally connected would be assigned the trivial group. We show by example that this hope is not realized.

3.1 Definition: If $\mathcal{U}$ is an open cover of X and if each element of $\mathcal{U}$ is a connected set, then we call $\mathcal{U}$ an open connected cover of X. We let $C^p(X)$ be as before and define $C_0^p(X) = \{\varphi | \varphi \in C^p(X)$ and there exists an open connected cover $\mathcal{U}$ of X with $\varphi = 0$ on $\mathcal{U}^{(p+1)}\}$.

Remark: Throughout this chapter all spaces under

consideration are assumed to have the property for open connected sets postulated above for X. It should be clear that this property is sufficient to guarantee that $C_0^p(X)$ is a group.

3.2 Definition: Let $Z^p(X) = \bar{\delta}^{-1}[C_0^{p+1}(X)]$ and $p \geq 0$, and $B^p(X) = C_0^p(X) + \bar{\delta}[C^{p-1}(X)]$, $p \geq 1$; $B^0(X) = \{0\}$.

3.3 Theorem: If $f: X \longrightarrow Y$ is such that the inverse image of an open connected set is open and connected then:

1) $f^{\#}[C_0^p(Y)] \subset C_0^p(X)$
2) $\bar{\delta}[C_0^p(X)] \subset C_0^{p+1}(X)$
3) $f^{\#}[Z^p(Y)] \subset Z^p(X)$
4) $f^{\#}[B^p(Y)] \subset B^p(X)$

Proof: 1) Let $\varphi \in C_0^p(Y)$, then there exists an open connected cover $\mathcal{V}$ of Y such that $\varphi = 0$ on $\mathcal{V}^{(p+1)}$. Consider $\{f^{-1}(v) | v \in \mathcal{V}\}$, by assumption this is an open connected cover of X and if $(x_0,\ldots,x_p) \in [f^{-1}(v)]^{p+1}$ then $[f^{\#}\varphi](x_0,\ldots,x_p) = \varphi(f[x_0],\ldots,f[x_p] \in \varphi(v^{p+1}) = 0$. Thus $f^{\#}\varphi \in C_0^p(X)$.

2) Let $\varphi \in C_0^p(X)$ and $\mathcal{U}$ be an open connected cover of X such that $\varphi = 0$ on $\mathcal{U}^{(p+1)}$. Let $(x_0,\ldots,x_{p+1}) \in u^{p+2}$, then $[\bar{\delta}\varphi](x_0,\ldots,x_{p+1}) = \sum_{i=0}^{p+1}(-1)^i \varphi(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) = 0$ since $(x_0,\ldots,\hat{x}_i,\ldots,x_{p+1}) \in u^{p+1}$ implies $\varphi(x_0,\ldots,\bar{x}_i,\ldots,x_{p+1}) = 0$. Thus $\bar{\delta}\varphi \in C_0^{p+1}(X)$.

3) Let $\varphi \in Z^p(Y) = \bar{\delta}^{-1}[C_0^{p+1}(Y)]$, then $\bar{\delta}\varphi \in C_0^{p+1}(Y)$ hence $f^{\#}\bar{\delta}\varphi \in f^{\#}[C_0^{p+1}(Y)] \subset C_0^{p+1}(X)$. Now $f^{\#}\bar{\delta}\varphi = \bar{\delta}f^{\#}\varphi$ and so $f^{\#}\varphi \in \bar{\delta}^{-1}[C_0^{p+1}(X)] = Z^p(X)$.

4) Let $\varphi \in B^p(Y) = C_0^p(Y) + \bar{\delta}[C^{p-1}(Y)]$, then $f^{\#}\varphi \in f^{\#}[C_0^p(Y)] + f^{\#}\bar{\delta}[C^{p-1}(Y)] \subset C_0^p(X) + \bar{\delta}[C^{p-1}(X)] = B^p(X)$, $p \geq 1$. The case $p = 0$ is clear.

3.4 <u>Definition</u>: $H^p(X) = Z^p(X)/B^p(X)$.

3.5 <u>Theorem</u>: Let $f\colon X \longrightarrow Y$ be such that the inverse image of an open connected set is open and connected, then there exists a unique homomorphism $f^*\colon H^p(Y) \longrightarrow H^p(X)$ such that $\alpha f^{\#} = f^*\beta$, where $\alpha\colon Z^p(X) \longrightarrow H^p(X)$ and $\beta\colon Z^p(Y) \longrightarrow H^p(Y)$ are the natural homomorphisms.

<u>Proof</u>: Induced Homomorphism Theorem.

3.6 <u>Theorem</u>: If $i\colon X \subset X$, then $i^*\colon H^p(X) \subset H^p(X)$.

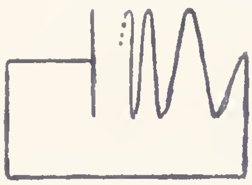
<u>Proof</u>: Let $h \in H^p(X)$ and $\varphi \in Z^p(X)$ with $\alpha\varphi = h$, then $i^*h = i^*\alpha\varphi = \alpha i^{\#}\varphi = \alpha\varphi = h$.

3.7 <u>Theorem</u>: If $f\colon X \longrightarrow Y$, $g\colon Y \longrightarrow Z$ are such that the inverse image of open connected sets are open and connected then $(gf)^* = g^*f^*\colon H^p(Z) \longrightarrow H^p(Z) \longrightarrow H^p(X)$.

<u>Proof</u>: We already know from Chapter II, that $(gf)^{\#} = f^{\#}g^{\#}$. By part three of Theorem 3.3 letting $\alpha\colon Z^p(X) \longrightarrow H^p(X)$, $\beta\colon Z^p(Y) \longrightarrow H^p(Y)$ and $\gamma\colon Z^p(Z) \longrightarrow H^p(Z)$ we may compute as follows: If $h \in H^p(Z)$ and $\varphi \in Z^p(Z)$ such that $\gamma\varphi = h$ then $f^*g^*(h) = f^*g^*(\gamma\varphi) = f^*\beta(g^{\#}\varphi) = \alpha f^{\#}(g^{\#}\varphi) = \alpha(gf)^{\#}\varphi = (gf)^*\gamma\varphi = (gf)^*(h)$.

3.8 <u>Example</u>: Let $X = A \cup B \cup C \cup D$ where $A = \{(0,y) | -1 \leq y \leq 1\}$, $B = \{(x, \sin(x^{-1})) | 0 < x \leq \pi^{-1}\}$,

$C = \{(x,0) \mid -1 \leq x \leq 0\}$ and $D = \{(x,y) \mid x = -1$ and $-2 \leq y \leq 0$, or $y = -2$ and $-1 \leq x \leq \pi^{-1}$, or $x = \pi^{-1}$ and $-2 \leq y \leq 0\}$, (see figure 1). Let $R \subset X \times X$ be defined by $R = (A \times A) \cup D(X^2)$ and let $Y = X/R$. It is easy to see that the removal of any pair of points from Y disconnects Y, and hence Y is topologically the unit circle. Since the definition given in this chapter is equivalent to that given in Wallace's notes, for locally connected spaces, we know that $H^1(Y) \cong G$.



(Fig. 1)

Let $f\colon X \longrightarrow X/R$ be the natural map. Then since X is compact and f is monotone and continuous we have that the inverse image of an open connected set is open and connected. Define $g\colon Y \longrightarrow X$ by the equations $g[f(x)] = x$ if $x \notin A$ and $g[f(A)] = (0,0)$. We note that if u is an open connected subset of X not containing $(0,0)$ then u-A is open and connected. To see that the inverse image under g of an open connected set u is open and connected we consider two cases.

1) If $(0,0) \notin u$, then $g^{-1}(u) = f(u-A)$ is open and connected since f restricted to X-A is a homeomorphism.

2) If $(0,0) \in u$, then $g^{-1}(u) = f(u)$ is connected since f is continuous and is open if there is an open set about f(A) contained in $g^{-1}(u)$. Now $(0,0) \in u$ implies there

exists $t \in (-1,0)$ and $z \in (0,\pi^{-1})$ such that $(t,0) \times \{0\} = P \subset u$ and $\{(x,\sin x^{-1}) | 0 < x < z\} = Q \subset u$. Letting $v = P \cup Q \cup \{(0,0)\}$ we have $f^{-1}f(v) = A \cup P \cup Q$ which is open in X and hence $f(A) \subset f(v) \subset f(a) = g^{-1}(u)$ implies $g^{-1}(u)$ is open.

Since $fg\colon Y \subset Y$, and X and Y satisfy the relevent hypotheses, we use theorems 3.5 and 3.6 to assert $(fg)^* = g^*f^*\colon H^p(Y) \subset H^p(X)$ and consequently, if $h_1, h_2 \in H^p(Y)$ with $f^*(h_1) = f^*(h_2)$, then $h_1 = g^*f^*(h_1) = g^*f^*(h_2) = h_2$ and f^* is a monomorphism. Thus $H^1(X)$ is non-trivial, while X is not locally connected.

CHAPTER IV

FINAL COMMENTS AND QUESTIONS

We first show by example that the cohomology groups as defined in Wallace's notes are different from those defined in Chapter II.

4.1 <u>Example</u>: Let G be an additive abelian group with the property that for each g in G there exists a positive integer n, depending on g, such that $ng = 0$. Suppose also that G is not nilpotent, i.e. there does not exist a positive integer n such that $ng = 0$ for all g in G. Letting $\underline{H}^p(X)$ denote the p-th cohomology group for the space X, computed with respect to G, as defined in [8], and recalling that $\underline{H}^o(X)$ is isomorphic to the group of functions mapping X into G which are continuous in the discrete topology of G, we easily show that $\underline{H}^o(X)$ is not isomorphic to $H^o(X)$ for $X = G$. For by theorem 2.9, $\varphi \in Z^o(X) \cong H^o(X)$ if and only if $\varphi: X \longrightarrow G$ is continuous in the discrete topology of G and $\{\varphi^{-1}(g) | g \in G\}$ is finite. Let $\{\varphi^{-1}(g) | g \in G\} = \{\varphi^{-1}(g_i) | 1 \leq i \leq n\}$ and let p_i be such positive integers that $p_i g_i = 0$, $1 \leq i \leq n$, and define $p = p_1 \times p_2 \times \ldots \times p_n$. Then $p\varphi = 0 \in Z^o(X)$ and hence each

element in $H^{o}(X)$ has finite order. But it is clear that $i\colon G \subseteq G$ is continuous in the discrete topology and has order zero, hence $\underline{H}^{o}(X)$ contains an element which is not of finite order.

Now Haskel Cohen in [2] gave a definition of co-dimension, based on the groups of [8], for locally compact spaces, and we have just seen that on locally compact spaces the groups of [8] differ from those of this thesis. It is natural to ask then, whether a definition similar to Cohen's will yield a dimension theory and if so, will this theory differ from Cohen's. Without reproducing the technical definitions we remark that the dimension of a space was defined in terms of the cohomological structure of its compact subspaces, and since the two cohomology theories involved agree on compact spaces it turns out that the aforementioned " similar" definition is, in fact, identical with that of Cohen's.

We have shown in Chapter II that we have constructed a cohomology in the sense of Mac Lane [5] but have not demonstrated that we have a cohomology theory in the sense of Eilenburg and Steenrod. Indeed, we have failed to verify the " Homotopy Axiom" of [3]. Although Lemmas 2.23 and 2.19, in the presence of the Reduction Theorem, were shown by Keesee in [4] to imply the Homotopy Axiom for compact pairs, there is no such short cut for the general case. We are unable, however, to produce an appropriate counterexample and hence must leave the question open. Because the Čech theory

based on finite coverings fails to satisfy this axiom we are led to conjecture that the axion also fails for the theory of Chapter II.

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BIOGRAPHICAL SKETCH

Marcus Mott McWaters Jr. was born January 21, 1939, in Little Rock, Arkansas. He was raised in New Orleans and Metairie, Louisiana, and was graduated in August, 1956, from East Jefferson High School.

In September, 1958, he entered Louisiana State University in New Orleans and was awarded the degree of Bachelor of Science from this institution in June, 1962. He then became a graduate student at Louisiana State Univeristy for one year, after which time he transferred to the University of Florida. From September, 1963, when he entered the University of Florida, he has worked as a graduate assistant in the Department of Mathematics and pursued his work toward the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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